2010/10

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DISCUSSION PAPER

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March 2010

Abstract

This paper introduces the notion of nested best response potentials for complete information games. It is shown that a unique maximizer of such a potential is a Nash equilibrium that is robust to incomplete information in the sense of Kajii and Morris (1997, mimeo).

Keywords: incomplete information, potential games, robustness, refinements.

JEL Classification: C72, C73

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This note is based on Chapter 5 of my Ph.D thesis submitted to Osaka University. I am grateful to my supervisors Masaki Aoyagi and Atsushi Kajii for their instruction and encouragement. I also thank Julio D ́avila, Michihiro Kandori, Daisuke Oyama, and Takashi Ui, and the participants at the 3rd World Congress of the Game Theory Society (Northwestern University) and the Fall 2007 meeting of Japanese Economic Association (Nihon University) for helpful comments. This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Grant-in-Aid for 21st Century COE Program "Interfaces for Advanced Economic Analysis".

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the author.

1 Introduction

When analyzing a strategic situation, we often assume the common knowledge of payoffs and describe the situation as a complete information game. However, the predictions derived from a complete information game may be very different from those derived from games with an information structure that departs 'slightly' from common knowledge, as demonstrated by Rubinstein (1989) and Carlsson and van Damme (1993).

In this light, Kajii and Morris (1997a,b) introduced the concept of equilibria that are robust to incomplete information. A Nash equilibrium of a complete information game is said to be robust to incomplete information if every incomplete information game the payoffs of which differ from those of the original game only very rarely has a Bayesian Nash equilibrium close to the Nash equilibrium.

The robustness of equilibria in Kajii and Morris (1997a) and that in Kajii and Morris (1997b) are slightly different. In the definition of robustness, Kajii and Morris (1997a) considered incomplete information games such that, with high probability, each player knows that her payoffs coincide with those of the complete information game, and with low probability, payoffs may be quite different. Kajii and Morris (1997b) restricted attention to incomplete information games such that, with high probability, each player knows that her payoffs also coincide with those of the complete information game, but with low probability, she has some dominant action. Thus, if a Nash equilibrium is robust in the sense of Kajii and Morris (1997a) then it is also robust in the sense of Kajii and Morris (1997b). Whether or not the converse holds is an open question.

This paper provides a new sufficient condition for the robustness of an equilibrium to incomplete information in sense of Kajii and Morris (1997b) in terms of nested best response potential maximizers. The nested best response potential functions generalize the best response potential functions introduced in Morris and Ui (2005), applying the idea of 'nesting' based on Uno (2007). A best response potential function of a game is a real-valued function on the set of action profiles of the game that 'incorporates information' about every players' best response. It is known that every maximizer of a best response potential function is a Nash equilibrium of the game. It is as if the best response potential function is the payoff function of a representative agent that chooses strategies for all players.

In considering a nested best response potential function, we think of a representative agent for a subset T of players, instead of one for all of them: for each player i in T, given any belief over the set of strategy profiles of other players, maximizing this representative agent's payoff f_T yields a best response for each player i in T. Suppose that there is a partition $\mathcal{T}$ of players such that, for each member T of $\mathcal{T}$, there is such a representative agent whose payoff function is f_T.[3] Then the collection of f_T's can be seen as a new complete information game, where each member T in $\mathcal{T}$ is regarded as a single player. That is, the original game is reduced to a game with a smaller number of players.

Notice that such reduction can be nested: the new game among step 1 representative agents may be reduced to a game with an even smaller number of players, by considering a step 2 representative agent for each member of a partition of step 1 representative agents, and then a representative agent for each member of a partition of these, and so on. We say that a game has a nested best response potential if a game is reduced to a game with one representative agent through this process. We call a unique maximizer of nested best response potential function a nested best response potential maximizer.

The main result of this paper shows that a nested best response maximizer is robust to incomplete information in sence of Kajii and Morris (1997b) (Theorem 4.1).

In the literature, various sufficient conditions are given for robustness to incomplete information in sence of Kajii and Morris (1997b). Kajii and Morris (1997a,b) provided sufficient conditions in terms of unique correlated equilibria or **p**-dominance equilibria. Ui (2001) provided a sufficient condition based on an exact potential maximizer introduced by Monderer and Shapley (1996). Morris and Ui (2005) provided a sufficient condition in terms of generalized potential maximizers, which unified and strictly generalized the conditions of Kajii and Moris (1997a,b) and Ui (2001). Morris and Ui (2005) also introduced three special but tractable classes of generalized potentials: best response potentials, monotone potentials, and local potentials. Tercieux (2006) provided a sufficient condition for games with **p**-best response sets, which strictly generalized the conditions of Kajii and Morris (1997a,b) but specialized the condition in terms of local potential maximizers provided by Morris and Ui (2005). Oyama and Tercieux (2009) provided a sufficient condition for games with an iterated monotone potential maximizer, which generalized the condition in terms of monotone potential maximizers provided by Morris and Ui (2005).[4]

Our condition in terms of nested best response potential maximizers strictly generalizes the conditions in terms of exact potential maximizers in Ui (2000) and best response potential maximizers in Morris and Ui (2005). We demonstrate this point in Example 5.2 below. Moreover, we also demonstrate that our condition neither implies nor is implied by the conditions in terms of unique correlated equilibria or **p**-dominance equilibria provided by Kajii and Morris (1997a,b) and the condition in terms of **p**-best response sets provided

[3]This idea also has appeared as q-potential in Monderer (2007).

[4]In these conditions, the conditions excluding that in terms of exact potential maximizers provided by Ui (2000) and that in terms of best response potential maximizers provided by Morris and Ui (2005) also are sufficient conditions for robustness to incomplete information in sense of Kajii and Morris (1997a).

by Tercieux (2006). However, it is left as an open question whether our condition has some inclusion relation with the conditions in terms of generalized potential maximizers, the monotone potential maximizers, and local potential maximizers provided by Morris and Ui (2005), and the condition in terms of the iterated monotone potential maximizers provided by Oyama and Tercieux (2009).

The organization of this paper is as follows. Section 2 defines robust equilibria. Section 3 introduces the nested best response potentials. Section 4 provides the main result. Section 5 includes a discussion on the relations between the related literature and our result, and some concluding remarks.

2 Robust equilibria

A finite complete information game consists of a finite player set $N = \{1, \dots, n\}$, a finite action set A_i for $i \in N$, and the payoff function $g_i : A \to \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a complete information game $(N, (A_i)_{i\in N}, (g_i)_{i\in N})$ simply by $\mathbf{g}^N := (g_i)_{i\in N}$. For notational convenience, we write $a = (a_i)_{i\in N} \in A$; for $i \in N$, $A_{-i} = \prod_{j\neq i} A_j$ and $a_{-i} = (a_j)_{j\neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i\in T} A_i$, $a_T = (a_i)_{i\in T} \in A_T$, $A_{-T} = \prod_{i\in N\setminus T} A_i$, and $a_{-T} = (a_i)_{i\in N\setminus T} \in A_{-T}$. We write $(a_T, a_{-T}) \in A_T \times A_{-T}$. We write (a_i, a_{-i}) instead of $(a_{\{i\}}, a_{-\{i\}})$ for simplicity.

Consider an incomplete information game with the player set N and the set A of action profiles. Let Θ_i be a countable set of types of player i. The set of type profiles is $\Theta := \prod_{i\in N} \Theta_i$. We write $\Theta_{-i} = \prod_{j\neq i} \Theta_j$ and $\theta_{-i} = (\theta_j)_{j\neq i} \in \Theta_{-i}$; for $T \subseteq N$, $\Theta_T = \prod_{i\in T} \Theta_i$, $\theta_T = (t_i)_{i\in T} \in \Theta_T$, $\Theta_{-T} = \prod_{i\in N\setminus T} \Theta_i$, and $\theta_{-T} = (\theta_i)_{i\in N\setminus T} \in \Theta_{-T}$. Let $P \in \Delta(\Theta)$ be the common prior probability distribution over the set Θ of type profiles such that for each $i \in N$ and $\theta_i \in \Theta_i$, the marginal probability of θ_i is positive, i.e., $P_i(\theta_i) := \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i, \theta_{-i}) > 0$.[5] A payoff function for player i is a bounded function $u_i : A \times \Theta \to \mathbb{R}$. Since we will fix N, Θ, and A throughout the paper, we simply denote an incomplete information game by $(P, \mathbf{u})$, where $\mathbf{u} := (u_i)_{i\in N}$.

A strategy of player i is a function $\sigma_i : \Theta_i \to \Delta(A_i)$. We write Σ_i for the set of strategies of player i, and write $\Sigma = \prod_{i\in N} \Sigma_i$ and $\sigma = (\sigma_i)_{i\in N} \in \Sigma$; $\Sigma_{-i} = \prod_{j\neq i} \Sigma_j$ and $\sigma_{-i} = (\sigma_j)_{j\neq i} \in \Sigma_{-i}$; for $T \subseteq N$, $\Sigma_T = \prod_{i\in T} \Sigma_i$ and $\sigma_T = (\sigma_i)_{i\in T} \in \Sigma_T$. We write $\sigma_i(a_i|\theta_i)$ for the probability of action a_i given $\sigma_i \in \Sigma_i$ and $\theta_i \in \Theta_i$. For $\sigma \in \Sigma$, we write $\sigma(a|\theta) = \prod_{i\in N} \sigma_i(a_i|\theta_i)$ for $a \in A$ and $\theta \in \Theta$; for $\sigma_{-i} \in \Sigma_{-i}$, $\sigma_{-i}(a_{-i}|\theta_{-i}) = \prod_{j\neq i} \sigma_j(a_j|\theta_j)$ for $a_{-i} \in A_{-i}$ and $\theta_{-i} \in \Theta_{-i}$; for $T \subseteq N$ and $\sigma_T \in \Sigma_T$, $\sigma_T(a_T|\theta_T) = \prod_{i\in T} \sigma_i(a_i|\theta_i)$ for $a_T \in A_T$ and $\theta_T \in \Theta_T$.

[5]For a set S, $\Delta(S)$ denotes the set of all probability distributions over S.

A strategy profile $(\sigma_i)_{i\in N} \in \Sigma$ is a *(Bayesian Nash) equilibrium* of $(P, \mathbf{u})$ if, for each $i \in N$, and for each $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i)[\sum_{a\in A}\sigma(a|\theta_i,\theta_{-i})u_i(a,(\theta_i,\theta_{-i})) - \sum_{a_{-i}\in A_{-i}}\sigma_{-i}(a_{-i}|\theta_{-i})u_i((a_i', a_{-i}),(\theta_i,\theta_{-i}))] \geq 0$$

for all $a_i' \in A_i$, where $P(\theta_{-i}|\theta_i) = P(\theta_i,\theta_{-i})/\sum_{\hat{\theta}_{-i}\in\Theta_{-i}} P(\theta_i,\hat{\theta}_{-i})$.

Given a complete information game $\mathbf{g}^N$ and an incomplete information game $(P, \mathbf{u})$, for each $i \in N$, consider the subset $\bar{\Theta}_i$ of Θ_i such that, if $\theta_i \in \bar{\Theta}_i$ is realized, i's payoffs are given by g_i independently of the every types θ_{-i} of the other players:

$$\bar{\Theta}_i = \{\theta_i \in \Theta_i | u_i(a,(\theta_i,\theta_{-i})) = g_i(a) \text{ for all } a \in A, \theta_{-i} \in \Theta_{-i} \text{ with } P(\theta_i,\theta_{-i}) > 0\}. \qquad (1)$$

We write $\bar{\Theta} = \prod_{i\in N}\bar{\Theta}_i$. An incomplete information game $(P, \mathbf{u})$ is a δ-*elaboration* of $\mathbf{g}^N$ if $P(\bar{\Theta}) = 1 - \delta$, where $\delta \in [0, 1]$.

Kajii and Morris (1997a) introduced the robustness of equilibria to all elaborations.

Definition 2.1 An action distribution $\mu \in \Delta(A)$ is *robust to all elaborations* in $\mathbf{g}^N$ if, for any $\varepsilon > 0$, there exists $\bar{\delta} > 0$ such that, for any $0 < \delta \leq \bar{\delta}$, every δ-elaboration of $\mathbf{g}^N$ has an equilibrium σ with $\max_{a\in A}|\mu(a) - \sum_{\theta\in\Theta} P(\theta)\sigma(a|\theta)| \leq \varepsilon$.

Kajii and Morris (1997b) also introduced the following weaker notion of robustness of equilibria to 'canonical' elaborations.

A type $\theta_i \in \Theta_i \backslash \bar{\Theta}_i$ is *committed* if player i of this type has a strictly dominant action $a_i^{\theta_i} \in A_i$, i.e., $u_i((a_i^{\theta_i}, a_{-i}),(\theta_i,\theta_{-i})) > u_i((a_i, a_{-i}),(\theta_i,\theta_{-i}))$ for all $a_i \in A_i\backslash\{a_i^{\theta_i}\}$, $a_{-i} \in A_{-i}$, and $\theta_{-i} \in \Theta_{-i}$ with $P(\theta_i,\theta_{-i}) > 0$. A δ-elaboration $(P, \mathbf{u})$ of $\mathbf{g}^N$ is *canonical* if, for each $i \in N$, every $\theta_i \in \Theta_i\backslash\bar{\Theta}_i$ is a committed type.

Definition 2.2 An action distribution $\mu \in \Delta(A)$ is *robust to canonical elaborations* in $\mathbf{g}^N$ if, for every $\varepsilon > 0$, there exists $\bar{\delta} > 0$ such that, for all $0 < \delta \leq \bar{\delta}$, any canonical δ-elaboration of $\mathbf{g}^N$ has an equilibrium σ with $\max_{a\in A}|\mu(a) - \sum_{\theta\in\Theta} P(\theta)\sigma(a|\theta)| \leq \varepsilon$.

It is clear that if an action distribution is robust to all elaborations, then it is also robust to canonical elaborations.[6]

[6]Whether or not the converse holds is an open question.

3 Nested best response potentials

This section introduces the notion of nested best response potential for complete information games. The nested best response potentials generalize the best response potentials introduced in Morris and Ui (2005). A best response potential of a complete information game $\mathbf{g}^N$ is a real valued function f on the set A of action profiles such that, for each player i and for any i's belief $\lambda_i \in \Delta(A_{-i})$ over the set A_{-i} of other players' actions, i's best response against the belief λ_i in the alternative game where i's payoff function is given by f, is also his best response in the original game $\mathbf{g}^N$:[7]

Definition 3.1 (Morris and Ui, 2005) A function $f : A \to \mathbb{R}$ is a *best response potential* of $\mathbf{g}^N$ if, for each $i \in N$,

$$\arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) f(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \tag{2}$$

for all $\lambda_i \in \Delta(A_{-i})$.

We generalize the best response potentials by means of the 'nested construction' proposed in Uno (2007) as follows: firstly, for a partition $\mathcal{T}$ of N, we define the best response $\mathcal{T}$-potentials:[8]

Definition 3.2 Let $\mathcal{T}$ be a partition of N. A *best response $\mathcal{T}$-potential* of $\mathbf{g}^N$ is a 3-tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies that, for each $i \in T$,

$$\arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) f_T(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \tag{3}$$

for all $\lambda_i \in \Delta(A_{-i})$.

We denote such a best response $\mathcal{T}$-potential $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T\in\mathcal{T}}$ since action sets $(A_T)_{T\in\mathcal{T}}$ can be derived from the partition $\mathcal{T}$ of N and the set A of action

[7]There are three versions of best response potential in the literature. The best response potential of Morris and Ui (2005) is a cardinal version of the pseudo-potentials introduced in Dubey *et al.* (2006). The one of Morris and Ui (2004) is a version of best response potential where the inclusion of (2) is replaced by the equality. The one of Voorneveld (2000) is an ordinal version of best response potential of Morris and Ui (2004).

[8]The partition $\mathcal{T}$ best response potential generalizes Monderer (2007)'s q-potential: a strategic form game $\mathbf{g}^N$ has a q-potential if and only if $\mathbf{g}^N$ has a partition $\mathcal{T}$-potential, where q refers to the number of elements in $\mathcal{T}$ and the potentials in $(f_T)_{T\in\mathcal{T}}$ are meant to be the exact potentials in Monderer and Shapley (1996). If $\mathbf{g}^N$ is a q-potential game, then it has a partition $\mathcal{T}$ best response potential such that the number of elements of $\mathcal{T}$ is q. The converse is not true, since there is a best response potential game without an exact potential.

profiles in the original game $\mathbf{g}^N$. Notice that we can regard each best response $\mathcal{T}$-potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where $\mathcal{T}$ is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T; and for each $T \in \mathcal{T}$, f_T is the payoff function of T. The idea underlying the notion of the nested best response potentials is to construct such games iteratively:

Definition 3.3 A function $f : A \to \mathbb{R}$ is a *nested best response potential* of $\mathbf{g}^N$ if there exist a finite sequence $\{\mathcal{T}^k\}_{k=0}^K$ of partitions of N and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K = ((f_T^k)_{T \in \mathcal{T}^k})_{k=0}^K$ of 3-tuples such that

- $\{\mathcal{T}^k\}_{k=0}^K$ is a nested sequence of partitions of N: $\{\mathcal{T}^k\}_{k=0}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^0 = \{\{i\} | i \in N\}$ and $\mathcal{T}^K = \{N\}$;
- $\mathbf{f}^{\mathcal{T}^0} = (f_T^0)_{T \in \mathcal{T}^0}$ is the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^0(a) = g_i(a)$ for all $a \in A$;
- for each $k = 1, 2, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T \in \mathcal{T}^k}$ is a best response $\mathcal{T}^k$-potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T \in \mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each $T^k \in \mathcal{T}^k$ and for each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

$$\begin{aligned}
&\arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} \sum_{a_{-T^{k-1}} \in A_{-T^{k-1}}} \lambda_{T^{k-1}}(a_{-T^{k-1}}) f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) \\
&\subseteq \arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} \sum_{a_{-T^{k-1}} \in A_{-T^{k-1}}} \lambda_{T^{k-1}}(a_{-T^{k-1}}) f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}}) \qquad (4)
\end{aligned}$$

  for all $\lambda_{T^{k-1}} \in \Delta(A_{-T^{k-1}})$; and
- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

An action profile a^* is *a nested best response potential maximizer (NBRP-maximizer)* if $f(a^*) > f(a)$ for all $a \in A$ with $a \neq a^*$.

It is clear that if f is a best response potential of $\mathbf{g}^N$, then it is a nested best response potential of $\mathbf{g}^N$. However, even if a complete information game has a nested best response potential, it may not have a best response potential as shown in Example 3.4.

Example 3.4 Consider the three-person game $\mathbf{g}^{\{1,2,3\}} = (g_1, g_2, g_3)$ represented as Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

$\mathbf{g}^{\{1,2,3\}}$ does not have a best response potential. Indeed, note that $\mathbf{g}^{\{1,2,3\}}$ has a strict best response cycle $(U, C, T) \to (U, R, T) \to (U, R, B) \to (U, C, B) \to (U, C, T)$. Since games with a pseudo-potential cannot have strict best response cycles as shown by Kukushkin

T	L	C	R
U	$4,4,4$	$0,0,2$	$2,2,0$
M	$0,0,2$	$0,0,0$	$3,3,0$
D	$2,2,0$	$3,3,0$	$0,0,0$

B	L	C	R
U	$3,3,3$	$2,2,0$	$1,1,2$
M	$2,2,0$	$3,3,3$	$0,0,1$
D	$1,1,2$	$0,0,1$	$0,0,1$

Table 1: (g_1, g_2, g_3)

	UL	UC	UR	ML	MC	MR	DL	DC	DR
T	$4,4$	$2,0$	$0,2$	$2,0$	$0,0$	$0,3$	$0,2$	$0,3$	$0,0$
B	$3,3$	$0,2$	$2,1$	$0,2$	$3,3$	$1,0$	$2,1$	$1,0$	$1,0$

Table 2: $(f^1_{\{3\}}, f^1_{\{1,2\}})$

(2004), then games with a best response potential, which is a special case of pseudo-potentials, cannot have either. Thus $\mathbf{g}^{\{1,2,3\}}$ does not have a best response potential.

However, $\mathbf{g}^{\{1,2,3\}}$ has a nested best response potential. Indeed, $(f^1_{\{3\}}, f^1_{\{1,2\}})$ represented in Table 2 is a $\{\{3\},\{1,2\}\}$-best response potential of $\mathbf{g}^{\{1,2,3\}}$, where $f^1_{\{3\}}(\cdot) = g_3(\cdot)$ and $f^1_{\{1,2\}}(\cdot) = g_1(\cdot) = g_2(\cdot)$, and considering the best response $\{\{3\},\{1,2\}\}$-potential $(f^1_{\{3\}}, f^1_{\{1,2\}})$ as a two-person complete information game, we can show that $\mathbf{f}^{\{1,2,3\}} = (f)$ represented in Table 3 is a $\{\{1,2,3\}\}$-best response potential of $(f^1_{\{3\}}, f^1_{\{1,2\}})$. Thus $\mathbf{g}^{\{1,2,3\}}$ has a nested best response potential.

4 Nested potentials and robust equilibria

This section provides a sufficient condition for the robustness of equilibria in terms of the nested best response potential maximizers.

Theorem 4.1 *If $\mathbf{g}^N$ has a nested best response potential $f : A \to \mathbb{R}$ with a NBRP-maximizer a^*, then the action distribution $\mu \in \Delta(A)$ such that $\mu(a^*) = 1$ is robust to canonical elaborations in $\mathbf{g}^N$.*

	UL	UC	UR	ML	MC	MR	DL	DC	DR
T	4	2	0	2	0	0	0	0	0
B	3	0	2	0	3	1	2	1	1

Table 3: f

We can show this theorem by arguments similar to those of Theorem 6 in Morris and Ui (2005). Indeed, we replace Lemma 6 of Morris and Ui (2005) by Lemma 4.3 below. Let $(P, \mathbf{u})$ be a canonical δ-elaboration of $\mathbf{g}^N$ and consider the set of i's strategies of $(P, \mathbf{u})$ such that each committed type $\theta_i \in \Theta_i \backslash \bar{\Theta}_i$ chooses the strictly dominant action $a_i^{\theta_i}$:[9]

$$\Xi_i := \{\xi_i : \Theta_i \to A_i | \xi_i(\theta_i) = a_i^{\theta_i} \text{ for } \theta_i \in \Theta_i \backslash \bar{\Theta}_i\}.$$

Let $\Xi := \prod_{i \in N} \Xi_i = \{\xi : \Theta \to A | \xi(\theta) = (\xi_i(\theta_i))_{i \in N}$ for all $\theta \in \Theta,$ and $\xi_i \in \Xi_i$ for all $i \in N\}$. For $T \subseteq N$, $\Xi_T := \prod_{i \in N} \Xi_i$.

Note that if $(P, \mathbf{u})$ is canonical then Ξ is nonempty (Morris and Ui, 2005, Lemma 4).

Let $(P, \mathbf{u})$ be a canonical δ-elaboration of a complete information game $\mathbf{g}^N$ with a nested best response potential $f : A \to \mathbb{R}$. Define a function $V : \Xi \to \mathbb{R}$ such that

$$V(\xi) := \sum_{\theta \in \Theta} P(\theta) f(\xi(\theta))$$

for all $\xi \in \Xi$ and consider the set of its maximizers $\Xi^* := \arg\max_{\xi \in \Xi} V(\xi)$.

The function V is constructed by a similar way to that of generalized potentials in Morris and Ui (2005). We can show the following lemma by an argument similar to Lemma 5 in Morris and Ui (2005).

Lemma 4.2 *If Ξ is nonempty then Ξ^* is nonempty. If $\xi^* \in \Xi^*$ then*

$$\sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta) \geq 1 - \delta\kappa,$$

where κ is a positive constant.[10]

We show that there exists an equilibrium of $(P, \mathbf{u})$ assigning probability 1 to a maximizer $\xi^* \in \Xi^*$ of V, which corresponds to Lemma 6 in Morris and Ui (2005).

Lemma 4.3 *Suppose $\mathbf{g}^N$ has a nested best response potential f and $(P, \mathbf{u})$ is a canonical δ-elaboration of $\mathbf{g}^N$. For $\xi^* \in \Xi^*$, $(P, \mathbf{u})$ has an equilibrium $\sigma^* \in \Sigma$ such that $\sigma^*(\cdot|\theta)$ assigns probability 1 to the action $\xi^*(\theta)$ for all $\theta \in \Theta$, i.e., $\sigma^*(\xi^*(\theta)|\theta) = 1$ for all $\theta \in \Theta$.*

[9]Indeed, we set a domain $\mathcal{A}$ of generalized potential of Morris and Ui (2005) to $\prod_{i \in N}\{\{a_i\}|a_i \in A_i\}$.
[10]Or, $\kappa > 0$ is independent to δ. For example, $\kappa = [f(a^*) - \min_{a \in A} f(a)]/[f(a^*) - \max_{a \in A \backslash \{a^*\}} f(a)]$.

Proof. Let $\xi^* \in \Xi^*$. We want to show that, for each $i \in N$ and for each $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i)[u_i(\xi^*(\theta),(\theta_i,\theta_{-i})) - u_i((a_i,\xi^*_{-i}(\theta_{-i})),(\theta_i,\theta_{-i}))] \geq 0 \tag{5}$$

for all $a_i \in A_i$. Fix any $i \in N$ and $\theta_i \in \Theta_i$. If $\theta_i \in \Theta_i\backslash\bar{\Theta}_i$, then (5) is true, since $\xi^*_i(\theta_i)$ is the strictly dominant action $a_i^{\theta_i}$ of θ_i.

Suppose that $\theta_i \in \bar{\Theta}_i$. Let a positive integer K and sequences $(\mathbf{f}^{\mathcal{T}^k})_{k=0}^K$ and $(T^k)_{k=0}^K$ be such that, for each $k = 0,1,\dots,K$, $\{i\} = T^0 \subseteq T^1 \subseteq \cdots \subseteq T^{K-1} \subseteq T^K = N$, $T^k \in \mathcal{T}^k$ and, $f = f^K_{T^K}$, so that f is a nested best response potential.

Firstly, since $\xi^* \in \arg\max_{\xi\in\Xi}\sum_{\theta\in\Theta}P(\theta)f(\xi(\theta)) = \arg\max_{\xi\in\Xi}\sum_{\theta\in\Theta}P(\theta)f_{T^K}(\xi(\theta))$, we have

$$\sum_{\theta\in\Theta} P(\theta)[f^K_{T^K}(\xi^*(\theta)) - f^K_{T^K}(\xi_{T^{K-1}}(\theta_{T^{K-1}}),\xi^*_{-T^{K-1}}(\theta_{-T^{K-1}}))] \geq 0$$

for all $\xi_{T^{K-1}} \in \Xi_{T^{K-1}}$. It is equivalent to, for each $\theta_{T^{K-1}\backslash\{i\}} \in \Theta_{T^{K-1}\backslash\{i\}}$ with $P_{T^{K-1}}(\theta_i, \theta_{T^{K-1}\backslash\{i\}}) > 0$,

$$\sum_{\theta_{-T^{K-1}}\in\Theta_{-T^{K-1}}} P(\theta_{-T^{K-1}}|\theta_{T^{K-1}})[f^K_{T^K}(\xi^*(\theta)) - f^K_{T^K}(a_{T^{K-1}},\xi^*_{-T^{K-1}}(\theta_{-T^{K-1}}))] \geq 0$$

for all $a_{T^{K-1}} \in A_{T^{K-1}}$. Since $\mathbf{f}^{\mathcal{T}^K}$ is a best response $\mathcal{T}^K$-potential of $\mathbf{f}^{\mathcal{T}^{K-1}}$, by (4), for each $\theta_{T^{K-1}\backslash\{i\}} \in \Theta_{T^{K-1}\backslash\{i\}}$ with $P_{T^{K-1}}(\theta_i,\theta_{T^{K-1}\backslash\{i\}}) > 0$, we have

$$\sum_{\theta_{-T^{K-1}}\in\Theta_{-T^{K-1}}} P(\theta_{-T^{K-1}}|\theta_{T^{K-1}})[f^{K-1}_{T^{K-1}}(\xi^*(\theta)) - f^{K-1}_{T^{K-1}}(a_{T^{K-1}},\xi^*_{-T^{K-1}}(\theta_{-T^{K-1}}))] \geq 0 \tag{6}$$

for all $a_{T^{K-1}} \in A_{T^{K-1}}$.

Next, (6) is equivalent to

$$\sum_{\theta\in\Theta} P(\theta)[f^{K-1}_{T^{K-1}}(\xi^*(\theta)) - f^{K-1}_{T^{K-1}}(\xi_{T^{K-1}}(\theta_{T^{K-1}}),\xi^*_{-T^{K-1}}(\theta_{-T^{K-1}}))] \geq 0 \tag{7}$$

for all $\xi_{T^{K-1}} \in \Xi_{T^{K-1}}$. Since $T^{K-2} \subseteq T^{K-1}$, we have $T^{K-1} = T^{K-2} \cup T^{K-1}\backslash T^{K-2}$, and so $(\xi_{T^{K-2}},\xi^*_{T^{K-1}\backslash T^{K-2}}) \in \Xi_{T^{K-1}}$ for all $\xi_{T^{K-2}} \in \Xi_{T^{K-2}}$. Thus, (7) implies that

$$\sum_{\theta\in\Theta} P(\theta)[f^{K-1}_{T^{K-1}}(\xi^*(\theta)) - f^{K-1}_{T^{K-1}}(\xi_{T^{K-2}}(\theta_{T^{K-2}}),\xi^*_{-T^{K-2}}(\theta_{-T^{K-2}}))] \geq 0$$

for all $\xi_{T^{K-2}} \in \Xi_{T^{K-2}}$, where $\xi^*_{-T^{K-2}}(\theta_{-T^{K-2}}) = (\xi^*_{T^{K-1}\setminus T^{K-2}}(\theta_{T^{K-1}\setminus T^{K-2}}), \xi^*_{-T^{K-1}}(\theta_{-T^{K-1}}))$ for all $\theta_{-T^{K-2}} \in \Theta_{-T^{K-2}}$. By arguments similar to those given above, we have, for each $\theta_{T^{K-2}\setminus\{i\}} \in \Theta_{T^{K-2}\setminus\{i\}}$ with $P_{T^{K-2}}(\theta_i, \theta_{T^{K-2}\setminus\{i\}}) > 0$,

$$\sum_{\theta_{-T^{K-2}} \in \Theta_{-T^{K-2}}} P(\theta_{-T^{K-2}}|\theta_{T^{K-2}})[f^{K-2}_{T^{K-2}}(\xi^*(\theta)) - f^{K-2}_{T^{K-2}}(a_{T^{K-2}}, \xi^*_{-T^{K-2}}(\theta_{-T^{K-2}}))] \geq 0$$

for all $a_{T^{K-2}} \in A_{T^{K-2}}$.

By applying the arguments above to $K-3, K-4, \ldots, 0$, iteratively, we have

$$\sum_{\theta_{-T^0} \in \Theta_{-T^0}} P(\theta_{-T^0}|\theta_{T^0})[f^0_{T^0}(\xi^*(\theta)) - f^0_{T^0}(a_{T^0}, \xi^*_{-T^0}(\theta_{-T^0}))] \geq 0$$

for all $a_{T^0} \in A_{T^0}$. Since $T^0 = \{i\}$ and $f_{T^0} = g_i$, we have

$$\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i)[g_i(\xi^*(\theta)) - g_i(\xi_i(\theta_i), \xi^*_{-i}(\theta_{-i}))] \geq 0$$

for all $a_i \in A_i$. Since $\theta_i \in \bar{\Theta}_i$, we have (5). ∎

Lemma 4.2 and 4.3 imply that $(P, \mathbf{u})$ has an equilibrium $\sigma^* \in \Sigma$ such that $\sigma(\xi^*(\theta)|\theta) = 1$ for all $\theta \in \Theta$, where $\xi^* \in \Xi^*$, and

$$\begin{aligned} \sum_{\theta \in \Theta} P(\theta)\sigma^*(a^*|\theta) &\geq \sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta)\sigma^*(a^*|\theta) \\ &= \sum_{\theta \in \Theta, \xi^*(\theta) = a^*} P(\theta) \geq 1 - \delta\kappa. \end{aligned}$$

Thus, for each $\varepsilon > 0$, if we choose $\bar{\delta} = \varepsilon/\kappa > 0$, then, for each $\delta \leq \bar{\delta}$, every canonical δ-elaboration $(P, \mathbf{u})$ of $\mathbf{g}^N$ has an equilibrium σ^* such that $1 - \sum_{\theta \in \Theta} P(\theta)\sigma^*(a^*|\theta) \leq \varepsilon$, which completes the proof.

5 Concluding remarks: related literature

This paper introduces the notion of nested best response potential for complete information games and shows that a unique maximizer of a nested best response potential is a Nash equilibrium of the game that is robust to incomplete information.

The remaining shows the relationships between our sufficient condition (Theorem 4.1) and other sufficient condition in the literature. Morris and Ui (2005) provided a sufficient condition for games with a best response potential maximizer.[11]

Theorem 5.1 (Morris and Ui, 2005) *If* $\mathbf{g}^N$ *has a best response potential* $f : A \to \mathbb{R}$ *with a unique maximizer* a^**, then the action distribution* $\mu \in \Delta(A)$ *such that* $\mu(a^*) = 1$ *is robust to canonical elaborations in* $\mathbf{g}^N$*.*

Our condition strictly generalizes Theorem 5.1. That is, if a game has a unique maximizer of a best response potential then the game also has a unique maximizer of a nested best response potential since a best response potential is a nested best response potential. However, a game with a unique maximizer of a nested best response potential may not have a best response potential maximizer, which is shown by Example 5.2.

Example 5.2 (Example 3.4, continued) Consider the game $\mathbf{g}^{\{1,2,3\}}$ in Table 1 again. Theorem 5.1 does not apply to the game $\mathbf{g}^{\{1,2,3\}}$ since it does not have a best response potential, as demonstrated in Example 3.4. However, Theorem 4.1 applies to the game. Indeed, $\mathbf{g}^{\{1,2,3\}}$ has a nested best response potential f in Table 3. (U, L, T) is the unique maximizer of the potential f. Thus, by Theorem 4.1, the action distribution $\mu \in \Delta(A)$ such that $\mu(U, L, T) = 1$ is robust to canonical elaborations in $\mathbf{g}^{\{1,2,3\}}$.

Tercieux (2006) provided a sufficient condition that applies to games with the $\mathbf{p}$-best response set introduced by Tercieux (2004). For $S_{-i} \subseteq A_{-i}$, we denote $\Lambda^{p_i}(S_{-i})$ for the set of player i's beliefs such that the event that the other players play in S_{-i} has a probability greater than p_i:

$$\Lambda^{p_i}(S_{-i}) := \Big\{\lambda_i \in \Delta(A_{-i}) \big| \sum_{a_{-i} \in S_{-i}} \lambda_i(a_{-i}) \geq p_i\Big\}$$

A set of action profiles S is a $\mathbf{p}$-best response set if, when each player i believes with probability greater than p_i that the other players will play in S_i, player i has a best response in S_i:

Definition 5.3 (Tercieux, 2004) Let $\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1]^N$. A set $\prod_{i \in N} S_i$, $(S_i \subseteq A_i, i \in N)$ is a $\mathbf{p}$-*best response set* if, for each player $i \in N$, for all $\lambda_i \in \Lambda^{p_i}(S_{-i})$, there exists $a_i \in S_i$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a'_i, a_{-i})$$

for $a'_i \notin S_i$. A $\mathbf{p}$-best response set S is a *minimal* $\mathbf{p}$-*best response set* if no $\mathbf{p}$-best response set is a proper subset of S.

[11] Theorem 5.1 strictly generalized the sufficient condition in terms of exact potential maximizers by Ui (2001), which was demonstrated by Morris and Ui (2004).

For an action distribution $\mu \in \Delta(A)$, we denote the support of μ by $Supp(\mu) := \{a \in A | \mu(a) > 0\}$.

If there exists a unique correlated equilibrium μ which support is a $\mathbf{p}$-best response set with $\sum_{i \in N} p_i < 1$ then μ is robust to all elaborations in $\mathbf{g}^N$:[12]

Theorem 5.4 (Tercieux, 2006) *Let S be a $\mathbf{p}$-best response set with $\sum_{i \in N} p_i < 1$ of game $\mathbf{g}^N$. If there exists a unique correlated equilibrium μ such that $Supp(\mu) \subseteq S$, then μ is robust to all elaborations in $\mathbf{g}^N$.*

Theorem 5.4 and our condition have no including relation. Indeed, Theorem 5.4 does not apply to the game of Example 3.4.

Example 5.5 (Example 5.2, continued) Consider the game $\mathbf{g}^{\{1,2,3\}}$ in Table 1 again. As shown in Example 5.2, by Theorem 4.1, (U, L, T) is robust to canonical elaborations in $\mathbf{g}^{\{1,2,3\}}$. However, Theorem 5.4 does not apply to the game. Note that $S := \{U, M\} \times \{L, C\} \times \{T, B\}$ is a minimal (p_1, p_2, p_3)-best response set $\sum_{i \in N} p_i < 1$. Indeed, S is a (p_1, p_2, p_3)-best response set for $p_1, p_2 \geq 2/5$ and $p_3 \geq 0$; $\{(U, L, T)\}$ is a (p_1, p_2, p_3)-best response set for $p_1, p_2 \geq 3/5$ and $p_3 \geq 3/4$; $\{(M, C, B)\}$ which is a (p_1, p_2, p_3)-best response set for $p_1, p_2 \geq 1/2$ and $p_3 \geq 2/5$; $\{U\} \times \{L\} \times \{T, B\}$ is a (p_1, p_2, p_3)-best response set for $p_1, p_2 \geq 3/5$ and $p_3 \geq 0$; $\{U, M\} \times \{L, C\} \times \{T\}$ is a (p_1, p_2, p_3)-best response set for $p_1, p_2 \geq 2/5$ and $p_3 = 1$; and so on. And, we can show that there exist no unique correlated equilibrium μ such that $Supp(\mu) \subseteq S$, since $\mathbf{g}^{\{1,2,3\}}$ has multiple equilibria (U, L, T) and (M, C, B). Thus, Theorem 5.4 does not apply to the game.

On the other hand, Theorem 5.4 applies to the following game, as shown in Tercieux (2006), but our result does not apply to the example.[13]

Example 5.6 (Tercieux, 2006) Consider the two-person game $\mathbf{g}^{\{1,2,3\}} = (g_1, g_2)$ represented as Table 4, where player 1 chooses the row and player 2 chooses the column.

---

[12]Theorem 5.4 strictly generalized two sufficient conditions for the robustness to incomplete information provided by Kajii and Morris (1997a), i.e., the one in terms of unique correlated equilibria and the one in terms of $\mathbf{p}$-dominant equilibria with $\sum_{i \in N} p_i < 1$. Indeed, $A = \prod_{i \in N} A_i$ is always a $\mathbf{p}$-best response set with $\sum_{i \in N} p_i < 1$. Thus, if there exists a unique correlated equilibrium of $\mathbf{g}^N$, it is robust to all elaborations in $\mathbf{g}^N$ by Theorem 5.4. And, if a $\mathbf{p}$-best response set with $\sum_{i \in N} p_i < 1$ is a singleton $\{a\}$ then it is called $\mathbf{p}$-dominant equilibrium with $\sum_{i \in N} p_i < 1$. So, by Theorem 5.4, a is robust to all elaborations in $\mathbf{g}^N$.

[13]There are other examples to which Theorem 4.1 does not apply but Kajii and Morris' (1997a,b) conditions do. For example, two-person matching penny games have a unique correlated equilibrium, so the equilibrium is robust to all elaborations by a condition of Kajii and Morris (1997a,b). The game has neither a best-response potential nor a nested best-response potential, since the game is two-person game and has a best-response cycle. So, Theorem 4.1 does not apply to the game.

	L	C	R
U	$5,5$	$1,4$	$1,0$
M	$4,1$	$3,4$	$5,3$
D	$0,1$	$4,3$	$4,4$

Table 4: (g_1, g_2)

Theorem 5.4 applies to the game. The game $\mathbf{g}^{\{1,2\}}$ has a minimal (p_1, p_2, p_3)-best response set $\{M, D\} \times \{C, R\}$ with $p_1, p_2 = 1/3$, and $\mu \in \Delta(A)$ such that $\mu(a') = 1/4$ for $a' \in \{M, D\} \times \{C, R\}$ is a unique correlated equilibrium with support in $\{M, D\} \times \{C, R\}$, as shown in Tercieux (2006). By Theorem 5.4, μ is robust to all elaborations in $\mathbf{g}^{\{1,2\}}$.

However, Theorem 4.1 does not apply to the game. Indeed, note that $\mathbf{g}^{\{1,2\}}$ has a strict best response cycle $(M, C) \to (D, C) \to (D, R) \to (M, R) \to (M, C)$. Since games with a pseudo-potentials cannot have strict best response cycles, games with a best response potential, which is a special case of pseudo-potentials, cannot either. Thus, $\mathbf{g}^{\{1,2\}}$ does not have a best response potential. For two-person games, a best response potentials is equivalent to a nested best response potential. Thus, Theorem 4.1 does not apply to the game.

There are other sufficient conditions for the robustness to incomplete information. Morris and Ui (2005) provided a sufficient condition for games with generalized potential maximizer, which unified and generalized Kajii and Morris' (1997a,b) conditions and the condition in terms of exact potential maximizers by Ui (2001). Morris and Ui (2005) also introduced special but tractable classes of generalized potentials: monotone potentials and local potentials. Oyama and Tercieux (2009) provided a sufficient condition for games with an iterated monotone potential maximizer, which generalized the condition in terms of monotone potential maximizers.[14] The conditions in terms of monotone potential maximizers, local potential maximizers, and iterated monotone potential maximizers apply only to games with linearly ordered action sets. On the contrary, our condition in terms of nested best response potential maximizers applies to games with unordered action sets. Once we restrict attention to games with linearly ordered action sets, it remains to be checked what is the relation between the above conditions and ours.It also remains to be checked what is the relation between the condition in terms of generalized potential maximizers and ours.

[14]Whether or not the condition in terms of iterated monotone potential maximizers strictly generalizes the one in terms of monotone potential maximizers is an open question.

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